Optimum Vibration Isolation System Design Using Linear Programing

PETER A. ORLIN*
TRW Systems, Redondo Beach, Calif.

AND

MALCOLM A. CUTCHINS†
Auburn University, Auburn, Ala.

A method for the computer-aided design of a six degree-of-freedom vibration isolator system is presented. The method is predicated on the description of the desired mode shapes of the final system and a range of allowed frequencies for those modes. Use is made of the Kronecker or direct matrix product to restructure the system mass and stiffness matrices so that the elemental mass and stiffness properties are available as variables. A linear program is formed from these representations and the values of the elemental properties that are necessary to provide the desired mode shapes are determined. It is shown that the method is accurate and that the effects of viscous damping can be easily included. An example is given of a mounting system design.

Nomenclature‡

A = geometric stiffness
B = geometric damping
C = damping
D = double index to matrix operator
d_i = variables used in sweeping technique
G = modal stiffness
K = stiffness
L = modal damping
M = mass
Q = system quantity description
S_i = sweeping matrix
s = diagonal center of mass location
U = matrix to double index operator
Δ = difference in squares of frequency
ε = small quantity
ζ = damping ratio = C/C_{cr}
Λ = coordinate transforms (direction cosines)
ω = circular frequency
ω_D = damped circular frequency
ω_o = natural circular frequency
ϕ = mode shape
$\searrow$ = diagonal matrix or vector
$\bar{\ }$ = referenced to local coordinate
[ ] = matrix quantity
{ } = column vector
$\hat{\ }$ = doubled indexed quantity
$\hat{\mathbf{K}}$ = [$\searrow K \searrow$] put in columnar form
$\times$ = direct matrix product[7]
T = superscript for transpose of matrix
$\lfloor\ \rfloor$ = row vector

Subscripts

e = elemental quantity
i = property of the ith Mode
G = global referenced
M = median value
lm or pq = topological indices related to element
o = orthogonality related
R = reduced number of coordinates
r = property of the rth element
ω = frequency related
x, y, z = coordinate spatial variables
ξ, ψ, θ = coordinate angular variable about the x, y, and z axes

Received January 22, 1974; presented as Paper 74-266 at the AIAA 10th Annual Meeting and Technical Display, Washington, D.C., January 28–30, 1974; revision received May 13, 1974.

Index categories: Structural Design, Optimal; Computer Technology and Computer Simulation Techniques.

* Member of the Technical Staff; formerly Graduate Student, Auburn University. Member AIAA.

† Associate Professor of Aerospace Engineering. Member AIAA.

‡ The use of the carat mark to identify matrices transformed to vectors is necessitated by the mixed form of the equations and the requirement for maintaining identification of the source of the various quantities. While it is realized that this transformation of quantities from a matrix to a double indexed vector may be confusing at first, the authors have attempted to be consistent in its application.

I. Introduction

FREQUENTLY in the mounting and vibration isolation of equipment it is desirable that the predominant mode of vibration be of a specified shape and/or avoids certain frequency ranges. For instance, such desires may exist for aircraft or spacecraft mounted optical equipment where it is necessary to locate the predominant vibrational nodal point at a cardinal point of the objective lens system to minimize image blurring.

In most mechanical systems, it is desirable to avoid having natural frequencies in frequency ranges which might be excited by the environment to which the system is exposed and to satisfy limits on peak acceleration.

Although many approaches exist to the solution of design problems for specified vibration environments, they are either nonlinear,[1–3] or are not frequency-mode shape oriented.[4] Hence, the application of linear programing (LP)[5] as used in this paper brings several advantages to the solution of optimal design problems; the most important of which are the use of a finite number of iterations and the elimination of local optima. In addition, linear programing has been used for business applications for many years and in such use a sophisticated set of algorithms and computer codes have been developed. This is not generally the case for nonlinear approaches.

The method to be followed in this paper can be briefly stated as follows: first, it is assumed that a rigid body and its mounting geometry have been defined (Fig. 1), and that the desired mode shapes and natural frequencies (or frequency ranges) have been specified. Next, the specified vibration isolator locations will be used to form a matrix $[A]$, which, when multiplied by the isolator stiffnesses, will yield the structural $[K]$ matrix. Then, the range of values of the $[K]$ matrix will be limited by requiring that its components be orthogonal to the chosen mode shapes. Thus, the range of isolator stiffnesses, when used as variables in a linear program, will be restricted to those which are allowed by the chosen mode shapes and mounting

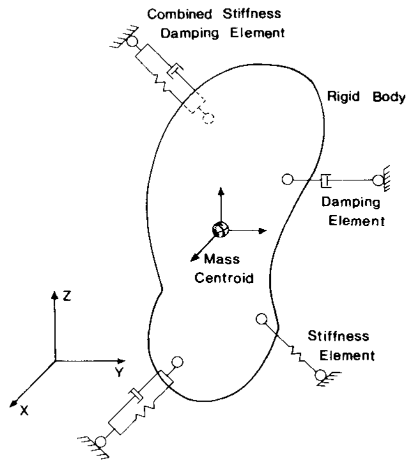

Fig. 1 General mounting arrangement.

geometry. It is important in this approach to use a double index notation which makes it possible to write rectangular matrices in vector form, and to use the direct matrix product which performs congruency transformations with a single ordinary matrix multiplication. Such a formulation of the problem preserves the linear relationship between the geometry of the isolator locations, the mode shape orthogonality requirements, and the elemental isolator stiffness values. Therefore, the problems associated with a nonlinear formulation are avoided and the full facilities of linear programing methods can be used to select from among the possible choices of isolator stiffnesses.

In addition to an optimization method involving $[K]$, consideration will be given to the problems associated with operating on the system damping matrix in a similar, but approximate, manner.

II. Theory

It is assumed that the reader is familiar with the conventional approach (forward problem) to the multiple degree of freedom vibration problem, both with and without damping. The *inverse problem* of proceeding, in the case of no damping, from a specified set of modes $[\phi]$ and frequencies $[\omega^2]$ to an optimal stiffness matrix is most easily presented using a double index notation.[7] In this notation, a quantity a_{ij} is the same as some single subscripted quantity except that the subscripts have been retained to indicate, for instance, the quantities' origin as an element in an array. An essential aspect of the notation is the nonunique transformation of a square array to a double indexed vector. For the purposes of this paper, this transformation will be defined as a stacking operation. For example,

$$\text{If} \qquad [F] = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \qquad \text{then} \qquad \hat{\mathbf{F}} = \begin{Bmatrix} f_{11} \\ f_{21} \\ f_{12} \\ f_{22} \end{Bmatrix} \tag{1}$$

The transformation can be represented symbolically by the operators $D(\ )$ and $U(\ )$ so that $\hat{\mathbf{F}} = U[F]$ and $[F] = D(\hat{\mathbf{F}})$. A more rigorous exposition is presented in Ref. 12.

Consider an arbitrary structure. A structural element of stiffness $[\bar{K}]$ in local coordinates can be rotated into global coordinates by a matrix of direction cosines Λ so that

$$[K_G] = [\Lambda]^T[\bar{K}][\Lambda] \tag{2}$$

Hence

$$\hat{\mathbf{K}}_G = ([\Lambda] \times [\Lambda])^T \hat{\bar{\mathbf{K}}} \tag{3}$$

where both $\hat{\mathbf{K}}_G$ and $\hat{\bar{\mathbf{K}}}$ are referred to the global indexing system. The vector $\hat{\bar{\mathbf{K}}}$ can be represented as

$$\hat{\bar{\mathbf{K}}} = \hat{\mathbf{Q}}\hat{\bar{K}} \tag{4}$$

where $\hat{\mathbf{Q}}$ is a system quantity description vector and $\hat{\bar{K}}$ is a scalar quantity with double indices. Following Martin,[8] these indices are used in a topological sense to represent the stiffness between two points in space, and not as positional indicators in an array. Since each scalar $\bar{K}$ corresponds to a unique elemental property connecting point i with point j, the double index notation is applicable.

Hence, using Eq. (3) in Eq. (4), it follows that, for n variable properties of any particular element

$$\hat{\mathbf{K}}_G = ([\Lambda] \times [\Lambda])_1{}^T\hat{\mathbf{Q}}_1\hat{\bar{K}} + \cdots + ([\Lambda] \times [\Lambda])_n{}^T\hat{\mathbf{Q}}_n\hat{\bar{K}}_n \tag{5}$$

Letting

$$\hat{\mathbf{A}}_r = ([\Lambda] \times [\Lambda])_r{}^T\hat{\mathbf{Q}}_r \tag{6}$$

describe the rth property of an element, the stiffness of element lm with an rth variable property can be written as

$$\{\hat{\mathbf{K}}_{G_{lm}}\}_r = \{\hat{\mathbf{A}}_{lm}\hat{\bar{K}}_{lm}\}_r \tag{7}$$

The system $[K]$ matrix can now be assembled from the $\hat{\mathbf{K}}_G$ submatrices since indices were numbered globally. This operation can be done formally through matrix multiplication, but as is pointed out in Ref. 9, the result is the same as adding element stiffnesses at corresponding positions in the system matrix. Hence, for the rth and the sth properties which contribute to element ij of the global $[K]$ matrix,

$$\hat{\mathbf{K}} = \begin{Bmatrix} \vdots & \vdots & \vdots \\ \{\hat{A}_{lm,ij}\hat{\bar{K}}_{lm,ij}\}_r + \{\hat{A}_{pq,ij}\hat{\bar{K}}_{pq,ij}\}_s + \cdots \\ \vdots & \vdots & \vdots \end{Bmatrix} \tag{8}$$

where there will be one row of sums for each ij pair. As can be seen from the form of $\hat{\mathbf{K}}$, it can be represented as the product of a square matrix and a vector containing the elemental $\hat{\bar{K}}$'s. Thus

$$\hat{\mathbf{K}} = [A]\hat{\mathbf{K}}_e \tag{9}$$

where $\hat{\mathbf{K}}_e$ is a column vector formed from the individual $\{\hat{\bar{K}}_{lm}\}_r$.

The matrix $[A]$ contains the terms describing the geometry of the space between points in the system, and the vector $\hat{\mathbf{K}}_e$ can be used as a variable in describing a family of $[K]$ which are related by $[A]$.

A formal discussion of the reduction of Eq. (9) for the case of certain bound nodes to reduced matrices is given in Ref. 12. Essentially, equilibrium conditions are applied and the fixed geometric relationships implied by the rigid body are exploited to bring about the reduction. The general ideas, however, parallel those of the forward problem and will not be addressed here. The reduced form of Eq. (9) is

$$\hat{\mathbf{K}}_R = [A]_R\hat{\mathbf{K}}_e \tag{10}$$

This is the fundamental equation for the optimization of the support characteristics for rigid bodies. Equation (10) separates the system stiffness into a term involving only the geometric relations between the reference point, the free and the bound nodes, and a term involving only the elemental stiffness. Thus a family of $\hat{\mathbf{K}}_R$ vectors can be described for a system geometry $[A]_R$ simply by varying the values of the terms of $\hat{\mathbf{K}}_e$. Since the mode shapes and natural frequencies are a function of $\hat{\mathbf{K}}_R$, the values of $\hat{\mathbf{K}}_e$ required to yield a desired $\hat{\mathbf{K}}_R$ (if it is feasible) can be found by forming a linear program with the rows of $[A]_R$ used as constraints.

The problem at this point in the inverse problem, however, is that the required values and allowable range of $\mathbf{K}_R$ are not immediately evident. Such information can be obtained by the specification of desired mode shapes for the vibration of the system. Since the mass matrix $[M]_R$ is assumed constant, mode shapes which are orthogonal to $[M]_R$ will also be orthogonal to $[K]_R$ for the proper values of $\hat{\mathbf{K}}_e$. These *candidate* eigenvectors can be used to provide the value and range information about $[K]_R$ if the orthogonality conditions are applied. Thus, if $[\phi]$ is a matrix of candidate eigenvectors, then the following relation must hold for orthogonality:

$$[\phi]^T[K]_R[\phi] = \lceil K \rfloor \tag{11}$$

Equation (11) and an analogous relation for mass can be restated in direct product form[12] as

$$([\phi] \times [\phi])^T \hat{\mathbf{K}}_R = \ulcorner\hat{\mathbf{K}}\lrcorner \tag{12}$$

and

$$([\phi] \times [\phi])^T \hat{\mathbf{M}}_R = \ulcorner\hat{\mathbf{M}}\lrcorner \tag{13}$$

where the diagonal notation is retained in referring to the double indexed vectors to emphasize their form.

The frequency equations for the candidate modes can be written as

$$\ulcorner\hat{\mathbf{K}}\lrcorner = [\omega_o{}^2]\ulcorner\hat{\mathbf{M}}\lrcorner \tag{14}$$

Therefore, using Eqs. (10) and (12)

$$([\phi] \times [\phi])^T [A]_R \hat{\mathbf{K}}_e = [\omega_o{}^2]\ulcorner\hat{\mathbf{M}}\lrcorner \tag{15}$$

Letting $([\phi] \times [\phi])^T [A]_R = [G]$, we have the initial form of the LP for determining $\hat{\mathbf{K}}_e$:

$$[G]\hat{\mathbf{K}}_e = [\omega_o{}^2]\ulcorner\hat{\mathbf{M}}\lrcorner \tag{16}$$

Since the mode shapes are specified, the generalized masses $\ulcorner\hat{\mathbf{M}}\lrcorner$ are known and the right-hand side can be computed for required frequencies. However, if it is desired to specify a range of acceptable values of the frequencies $\omega_i{}^2$, the equality can be replaced by a set of strict inequalities for the rows containing the $\omega_i{}^2$ and the generalized masses. The remaining rows, of necessity, must equal zero as this is the orthogonality requirement between the modes.

Damping Modifications and Final Formation of the LP

Because the orthogonality conditions provide an uncoupled system of equations, the damping may be applied to each of the frequency constraint equations separately if the reduced damping matrix $[C_R]$ is some linear combination of $[M_R]$ and $[K_R]$.[6] While this restriction reduces the scope of feasible solutions, the alternative of specifying limits to the elements of $\hat{\mathbf{C}}_R$ as in the case of $\hat{\mathbf{K}}_R$ does not seem to be possible in a reasonable way. Hence, the optimization problem will be set up to provide completely uncoupled modes by requiring the damping to be orthogonal.

As was done with $\ulcorner\hat{K}\lrcorner$, a series of matrix operations can be performed on elemental damping elements to yield a vector $\ulcorner\hat{\mathbf{C}}$. Thus, we can write

$$\ulcorner\hat{\mathbf{C}}\lrcorner = ([\phi] \times [\phi])^T [B]_R \ulcorner\hat{C}\lrcorner_e \tag{17}$$

where $[B]_R$ is analogous to $[A]_R$ for stiffness in Eq. (10).

It is convenient to let $[L] = ([\phi] \times [\phi])^T [B]_R$ so that

$$\ulcorner\hat{\mathbf{C}}\lrcorner = [L]\hat{\mathbf{C}}_e \tag{18}$$

The diagonality of $[\ulcorner K \lrcorner]$, $\ulcorner\hat{\mathbf{M}}\lrcorner$, and $[\ulcorner C \lrcorner]$ can now be exploited to separate the 36 equations (for a six degree-of-freedom case) of $\ulcorner\hat{\mathbf{K}}\lrcorner$, $\ulcorner\hat{\mathbf{M}}\lrcorner$, and $\ulcorner\hat{\mathbf{C}}\lrcorner$ into two groups. The diagonal $(i = j)$ components of each of the matrices are the elements of the six frequency determining equations, while the remainder are elements of 30 equations requiring the modal orthogonality. As such, these 30 equations must always equal zero regardless of the ondiagonal values. As a consequence, $[L]$ and $[G]$ may be rewritten (using row interchange) as

$$[L] = \begin{Bmatrix} [L_\omega] \\ \cdots \\ [L_o] \end{Bmatrix} \qquad [G] = \begin{Bmatrix} [G_\omega] \\ \cdots \\ [G_o] \end{Bmatrix} \tag{19}$$

so that

$$\hat{\mathbf{K}} = [G_\omega]\hat{\mathbf{K}}_e \qquad \hat{\mathbf{K}}_o = [G_o]\hat{\mathbf{K}}_e = 0$$

and

$$\hat{\mathbf{C}} = [L_\omega]\hat{\mathbf{C}}_e \qquad \hat{\mathbf{C}}_o = [L_o]\hat{\mathbf{C}}_e = 0 \tag{20}$$

The orthogonality equations can now be set aside from further consideration during the optimization. While they remain necessary elements of the LP, only their relative values on a row basis are important. Since these are fixed by geometry and choice of eigenvectors, they are fixed throughout the optimization.

A further simplification results from the symmetry of the matrices. Since the orthogonality equations exist in redundant pairs, only half of them need be retained as 21 unique equations to guarantee orthogonality. Thus 21 equations will comprise the LP.

Turning to the uncoupled frequency equations, we can write for the ith mode

$$\omega_{D_i} = \omega_{o_i}(1 - \zeta_i{}^2)^{1/2} \tag{21}$$

This can be expanded using the usual definitions of ω_o and ζ to yield

$$\omega_{D_i}{}^2 = (1/M_i)K_i - (1/4M_i{}^2)C_i{}^2 \tag{22}$$

Thus if the C^2 were obtainable in terms of a linear combination of the elemental $\hat{\mathbf{C}}_e$, it would be possible to use Eq. (22) as a constraint equation for each mode. Unfortunately, this is not possible, as can be seen by examining the form of $\ulcorner\hat{\mathbf{C}}^2$. This approach leads to

$$\ulcorner\hat{\mathbf{C}}\lrcorner = ([\phi] \times [\phi])^T([C]_R \times [C]_R)U([\phi][\phi]^T) \tag{23}$$

an expression which is nonlinear in $[C]_R$ and not amenable to solution with this type of approach. Rewriting Eq. (23) in terms of a vector $\hat{\mathbf{C}}_R$ complicates the situation. Thus the use of damped frequency constraints cannot be formulated directly as a linear program and, as such, is beyond the scope of this paper. The problem can be approached on a separable programing basis,[10] but with consequent increase in dimensionality and uncertainty in the result.

While direct calculation of the damped frequencies is not possible in the inverse problem, the system damping can still be controlled in a manner consistent with the optimization of $\hat{\mathbf{K}}_e$. This can be accomplished by using the relationship between mass, frequency, and damping of a normal mode

$$C_{cr_i} = 2M_i\omega_{o_i} \tag{24}$$

If the range of allowed frequencies for the ith mode is restricted, some median value ω_M can be chosen as representative of the values within that range. Hence

$$(2/\omega_{M_i})K_{\omega_i} = (\omega_{o_i}/\omega_{M_i})C_{cr_i} \tag{25}$$

As successive optimization attempts are made, the range of allowable ω_o can be narrowed about the median value ω_M, so that the ratio ω_o/ω_M approaches one. Thus, Eq. (25) can be used to supply a measure of the critical damping of the mode.

In terms of the damping ratio, the following constraint can be written for each mode:

$$2(\zeta_i/\omega_{M_i})K_{\omega_i} - C_{\omega_i} \geqq 0 \tag{26}$$

Here the ζ is some maximum damping ratio chosen by the user.

Reversing the direction of the inequality in Eq. (26) will require that the damping ratio be less than or equal to ζ_i. Thus, by using two constraints, a range of allowable values of C_{ω_i} can be established. An additional constraint of the form,

$$(2/\omega_{M_i})K_{\omega_i} - C_{\omega_i} \geqq \varepsilon_i \tag{27}$$

must also be written for each mode to insure that the mode is underdamped. The value of ε must be chosen on the basis of the allowable range of ω_o and the nearness to critical damping that is desired in the solution.

The constraint equations for optimizations involving a constant mass matrix can now be formed from Eqs. (16, 20, 26, and 27)

$$\begin{bmatrix} \left[\ulcorner \dfrac{2\zeta}{\omega} \lrcorner\right] \; [G_\omega] & -[L_\omega] \\ [\ulcorner M \lrcorner]^{-1}[G_\omega] & [0] \\ [G_o] & [0] \\ [0] & [L_o] \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{K}}_e \\ \cdots \\ \hat{\mathbf{C}}_e \end{Bmatrix} \begin{matrix} \geqq \\ \leqq \\ \geqq \\ \geqq \\ = \\ = \end{matrix} \begin{Bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \varepsilon \end{Bmatrix}_i \\ \omega^2 \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix} \tag{28}$$

Note that Eq. (16) is shown here in its inequality form. This set of equations, when coupled with a suitable objective function, will determine the optimal elemental stiffnesses and damping values for a rigid body on flexible supports.

Possible Objective Functions and Constraints

With the fundamental constraint equations now established, some possible objective functions can be examined. In most

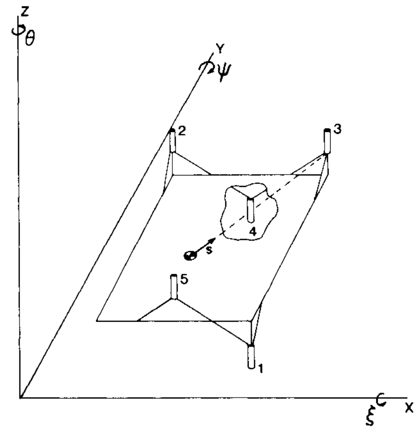

Fig. 2 Coordinates and supports for example rigid body.

cases, one constraint or a linear combination of similar constraints will be used. Thus, a particular component of $\hat{\mathbf{K}}$ or $\hat{\mathbf{C}}$ can be maximized or minimized to satisfy static requirements. It may also be desirable to attach costs or weights to each elemental property and minimize the system cost or weight.

Having thus chosen a suitable objective function and set of constraints, the optimization can be performed. Solving the forward problem at the end of each optimization attempt will yield the eigenvalues and eigenvectors of the modes. If it is necessary to control the order of the modes in terms of natural frequency, constraints can be added of the form

$$\left(\frac{1}{M_i}\lfloor G_\omega \rfloor_i - \frac{1}{M_j}\lfloor G_\omega \rfloor_j\right)\hat{\mathbf{K}}_e \geqq \Delta_{ij} \tag{29}$$

where Δ_{ij} is a desired difference of the square of the frequencies. It is also possible to form the constraints so as to require frequency multiples.

One last problem, that of determining the orthogonal candidate modes, can also be formulated as an LP. The approach is predicated on the *sweeping matrix* method.[6,11] Since the mass matrix is assumed constant, each candidate mode must be orthogonal to it. Thus, as modes are chosen, they can be used to sweep the mass matrix using methods in any of the several references[6,11] to yield the following equation:

$$[S_i]\mathbf{d}_i = \boldsymbol{\phi}_i \tag{30}$$

where $[S_i]$ is the swept mass matrix, $\boldsymbol{\phi}_i$ is the desired mode shape, and $\mathbf{d}_i$ is a vector of dummy variables required to remain finite during the optimization. If the equality is replaced by strict inequalities for the rows, then a range of possible values for each of the elements of $\boldsymbol{\phi}_i$ will be specified. To prevent unbounded solutions, the requirement that each element of the candidate modes $\boldsymbol{\phi}_i$ be in the range $-1 \leqq \phi_{ij} \leqq 1$ is similar to the requirement $\|\boldsymbol{\phi}_i\| \leqq 1$, but less restrictive.[10] Obviously, as each sweeping of the mass matrix $[M]$ is made, the choices for candidate modes $\boldsymbol{\phi}_i$ narrow, until the last mode is determined within a multiplicative constant.

III. Numerical Example

As an example of the application of the previous equations, a brief exposition is given here of the design of a set of isolators to support a rigid body with a variable location for its centroid. It is presented to show both the aptness of the method and nature of the optimal solution. Hence, the quantities involved are chosen for their simplicity so as to give the reader a basis for evaluating the results and not as a presentation of an actual physical system.

Figure 2 shows the arrangement of the isolator supports. Since the physical system is completely described by its mass, inertia matrix, mass centroid location, and the location of the mounting points, the shape of the supported item is unimportant and is only shown in outline. Five isolators, which have both damping and stiffness properties, will be selected by the linear program to maximize vertical stiffness (along the z axis). The modal matrix $[\phi]$ is chosen to be the identity matrix although the order of its columns will be dependent on the order of the frequencies determined by the LP. The mass matrix is chosen to be

$$\begin{bmatrix} 1 & 0 & 0 & & & \\ 0 & 1 & 0 & & 0 & \\ 0 & 0 & 1 & & & \\ & & & 0.1 & 0 & 0 \\ & 0 & & 0 & 0.1 & 0 \\ & & & 0 & 0 & 0.15 \end{bmatrix} \tag{31}$$

where the off diagonal submatrices are zero since the centroid has been chosen as the reference point.

Since the modal matrix has been chosen to be the identity matrix, the modes will be identified by the coordinate names associated with each axis. For the initial investigation of the feasibility of the mounting system, a set of "loose" frequency constraints are used. These values are given in Table 1.

Table 1

	Limiting Values				
	Frequency		Damping Ratio		
Mode	Low	High	Low	High	Remarks
x	5.0	15.0	0.1	1.0	Elemental values
y	10.0	33.0	0.1	1.0	$0.1 \leqq K_e \leqq 1000$ (lb/in.)
z	7.0	22.0	0.1	1.0	$10^{-6} \leqq C_e \leqq 100$ (lb-sec/in.)
ξ	50.0	200.0	0.5	0.8	All elements are pin jointed and
ψ	40.0	200.0	0.5	0.8	three elements (x, y, and z) are
θ	25.0	200.0	0.3	0.7	used to represent each isolator

The quantity which will be investigated during the optimization is the location of the centroid of the system measured by the parameter s where s is the normalized distance between a point at the center of the plate ($s = 0.0$) and mount 3 ($s = 1.0$). Here it is assumed that the mass to be mounted can be represented by an inertia ellipsoid with the properties of Eq. (31) which is capable of undergoing parallel displacements without rotation.

The results of the optimal design study are presented in Fig. 3 where the frequencies vs the parameter s are plotted. For the range of frequency and damping ratios chosen, the region of feasible solutions lies between $s = 0.1$ and $s = 0.5$. This region is limited on both sides by the orthogonality requirements between

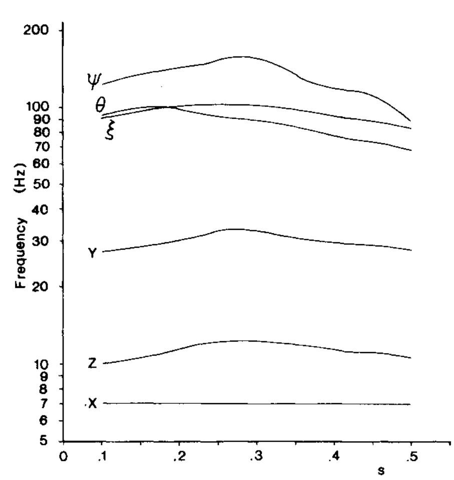

Fig. 3 Changes in frequencies as center of mass of rigid body is moved diagonally with s.

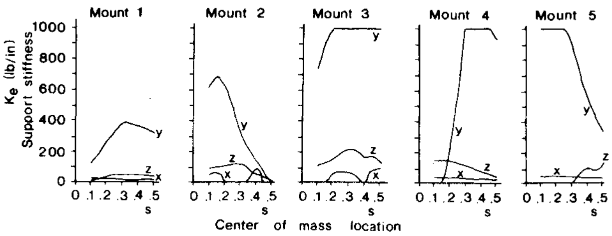

Fig. 4 Variation of 3 components of stiffness for optimum support as center of mass is moved diagonally (see Fig. 2 for mount locations).

the z and ξ modes. On the left side ($s = 0.1$) this limit is caused by mount 2 going to its extreme low value in stiffness in the y direction. The limitation on the right-hand side is caused by mount 2 going to its extreme low limit in damping. Within the feasible region, the z stiffness (K_{33}) rises to a maximum at about $s = 0.27$.

It should be noted that dependent on the particular range of constraints on the elemental values, and more importantly on the number of constraints involved in the LP, multiple optimal solutions exist. These are usually described as alternate optima and imply that when the objective function reaches its extremum the values of some of the elemental values may be arbitrary since they do not enter directly into the computation of the objective function. This is the case for the x and y elemental values in the example presented. The data shown is that which besides maximizing K_{33}, also resulted in keeping the damping in all modes at a minimum level. Additional constraints could have been formulated to force this result, but it did not prove necessary to do this. For systems with a large number of isolators this property of linear programing allows rapid identification of the elemental values which might be eliminated to simplify a design.

Figure 4 shows the values of the elemental stiffnesses for each isolator as s is varied. Note that the x stiffnesses of mounts 2 and 3 form a complementary pair since only one is required for the solution and the other is an alternate value. Consequently, either one of the x stiffness components could be eliminated without adversely affecting the solution.

IV. Discussion and Conclusions

At this point it is perhaps best to pause to evaluate this method in terms of the assumptions made and other published methods available.[1-4] Certainly, the requirements of small displacements and viscous damping limit the method to consideration of only those systems which can be suitably idealized. While these limitations prevent the inclusion of higher order damping and the limit cycle phenomena of large displacement oscillations, both of these design requirements are usually approached by first obtaining a linear solution. Thus, as a method for providing a first approximation to the nonlinear method, the method is quite suitable because of its rapid convergence and flexibility. This flexibility is due in part to the nonautomatic nature of the method. Since the designer is in over-all control of the optimization, his insight and experience can provide direction to the method, as opposed to many automatic schemes which depend on sometimes arbitrary selection rules.

The method is also open to the criticism that peak excursion and acceleration limits are not available as optimization parameters. This is a result of restricting the optimization to a linear form. While this formulation does allow consideration of damping ratios, the quadratic influence of the isolator damping in other quantities cannot be included. However, by using the linear approach to arrive at a preliminary design, it is then possible to expand Eq. (23) into a set of separable programing constraints which will allow inclusion of the quadratic influence of damping. Alternatively, a two step relinearization approach, such as the method of inscribed hyperspheres,[13] can be used. In either case, it is usually necessary to have an approximate linear solution valid in some local area before an attempt is made to include nonlinearities.

Finally, it is to be noted that the design of the vibration isolators, as such, is not included as part of the optimization. The method only provides values of damping and stiffness along a chosen set of axes for each isolator. It is felt that with the wide variety of isolators available, the choice of an isolator to meet the requirements of the linear approach should be left as a post-optimal problem.

Based on the analysis of the design of several support systems, the following conclusions can be drawn:

1) A knowledge of the system is required so that the candidate modes will be the ones that the system "wants" to have. Hence, the method is not intended for the untrained analyst.

2) The direction of optimization is best controlled by a human operator.

3) The method is rapid. Time to assemble the matrices, test candidate modes for possibility, perform the optimization, and output the actual eigenvalues is less than 2 min on an IBM 360/50,

4) The computer space required for the method is small, since the basic constraint matrix is of dimensions $21 \times (n_{e_k} + n_{e_c})$ and direct assembly of $[G]$ and $[L]$ obviates the need for actually forming any of the intermediate ($[\phi] \times [\phi]$) or $[A]$ matrices.

5) The method is ideally suited for small, interactive "smart" terminal applications.

6) The software for linear programing has been extensively developed, so that sophisticated diagnostic and postoptimal routines exist which further aid the design effort.

It is therefore felt that the method is a useful design tool in vibration engineering.

References

[1] Bell, R. W., "Retroversion from Preferred Eigenvalues to Determine Compatible Structural Characteristics," AIAA Paper 71-345, Anaheim, Calif., 1971.

[2] McCart, B., Haug, E., and Streeter, T., "Optimal Design of Structures with Constraints on Natural Frequency," *AIAA Journal*, Vol. 8, No. 6, June 1970, pp. 1012–1019.

[3] Seirig, A. and Howard, L., "An Approximate Normal Mode Method for Damped Lumped Parameter Systems," *Journal of Engineering, Inc.*, Vol. 89, No. 4, 1967, pp. 597–604.

[4] Turner, M., "Design of Minimum Mass Structures with Specified Natural Frequencies," *AIAA Journal*, Vol. 5, No. 3, March 1967, pp. 406–412.

[5] Hadley, G., *Linear Programming*, Addison-Wesley, Reading, Mass., 1962.

[6] Meirovitch, L., *Analytical Methods in Vibrations*, Macmillan, New York, 1967, p. 391.

[7] Pease, M., *Methods of Matrix Algebra*, Academic Press, New York, 1965, pp. 321–334.

[8] Martin, H., *Introduction to Matrix Methods of Structural Analysis*, McGraw-Hill, New York, 1966, pp. 29–60.

[9] Przemieniecki, J., *Theory of Matrix Structural Analysis*, McGraw-Hill, New York, 1968, p. 131.

[10] Hadley, G., *Nonlinear and Dynamic Programming*, Addison-Wesley, Reading, Mass., 1964, pp. 104–144.

[11] Vernon, J., *Linear Vibration and Control System Theory; with Computer Applications*, Wiley, New York, 1967, pp. 138–140.

[12] Orlin, P., "Optimum Vibration Isolation Design of Six Degree of Freedom Systems by use of Linear Programming," M.S. thesis, 1973, Auburn University, Auburn, Ala.

[13] Baldur, R., "Structural Optimization by Inscribed Hyperspheres," *Proceedings of the ASCE, Journal of the Engineering Mechanics Division*, Vol. EM3, June 1972, pp. 503–518.